A Study of the Finite Flat Plate Problem Using Various Kinetic and Continuum Models

A.N. Kudryavtsev, A.A. Shershnev and Ye.A. Bondar

Khristianovich Institute of Theoretical and Applied Mechanics, Russian Academy of Sciences, Siberian Branch, Novosibirsk 630090, Russia

Abstract. The supersonic flow past a finite flat plate at zero angle of attack is considered. The Navier-Stokes equations, Direct Simulation Monte Carlo (DSMC) statistical method and deterministic solving relaxation-type model kinetic equations using a high-order shock capturing scheme are used for numerical simulation of the problem. The results obtained with different approaches are compared and analyzed. The limits of continuum approach applicability as well as the accuracy of description based on model kinetic equations are studied.

Keywords: Kinetic models, finite-difference scheme, classical problem
PACS: 47.11.Bc, 47.40.Ki, 47.45.Ab

INTRODUCTION

It is well known that continuum description of gas flows fails when the molecular mean free-path length λ becomes comparable with the characteristic scale of the problem L. It happens if the Knudsen number $Kn = \lambda/L$ exceeds approximately 0.1. However, for the flows past bodies with sharp leading edges, restrictions are even more severe because in the immediate vicinity of the leading edge rarefaction effects are of great importance regardless of how small the Knudsen number based on the overall body size. Interest to such rarefied flows is caused by diverse problems of high-altitude aerodynamics, vacuum technology and, in recent years, fast-growing field of micro- and nanoelectromechanical systems (MEMS and NEMS). Numerical simulations of rarefied flows should be based on the kinetic description and nowadays the most widespread technique of simulation is statistical solution of the Boltzmann equation using the Direct Simulation Monte Carlo (DSMC) method. However, statistical scattering inherent in this method makes it less efficient when computing unsteady and/or low velocity flows. An alternative approach is based on deterministic solving the Boltzmann equation. However, since computation of the collision term in the Boltzmann equation requires evaluation of multidimensional integrals, for problems of practical interest it is prohibitively expensive even using modern supercomputers. The required computer resources can be significantly reduced if the collision integral is replaced by a simple relaxation-type term. Different models for the integral term were proposed, in particular well known Bhatnagar-Gross-Krook (BGK) model [1] and its modifications: the ellipsoidal statistical model (ES-BGK) of Holway [2] and the S-model of Shakhov [3].

Although these models were used to investigate many problems of rarefied gas dynamics, *a priori* their accuracy for flows which are far from equilibrium is not evident. A major drawback of all the model equations is that in accordance with them (and in contrast with the original Boltzmann equation) the probability of collision between two molecules does not depend on the their relative velocity. A purpose of the current paper is to investigate the accuracy of the relaxation-type kinetic model on a simple example of the strongly non-equilibrium flow past a finite flat plate with sharp leading and trailing edges immersed in a supersonic or hypersonic flow.

The low-density flow past a finite flat plate at zero angle of attack is one of the classical problems of rarefied gas dynamics. It has been studied both numerically and experimentally by many researchers. In particular, it has also been simulated using the BGK equation [4], the ES-BGK model [5] and the S-model [6]. More recently this problem was investigated in [7]. Detailed comparisons of continuum and kinetic solution were also presented in [8, 9].

In our simulations it is assumed that a infinitely thin plate of the unit length is aligned with the direction of the external supersonic flow. All computations were made for the hard sphere gas (the viscosity $\mu \sim T^{0.5}$). The Knudsen number Kn based on the plate length is 0.01 and the free-stream Mach number M_∞ is equal to 2 and 10. The free-stream temperature is taken to be equal to that of the plate surface, i.e. $T_w = T_\infty$, which corresponds approximately to the flight conditions when a vehicle surface is cooled. The results are compared with data of DSMC computations which are used as a reference solution and also with Navier-Stokes solutions.

NUMERICAL APPROACHES

Model kinetic equations

The two-dimensional relaxation-type kinetic equation in general form can be written as follows

$$\frac{\partial f}{\partial t} + \xi_1 \frac{\partial f}{\partial x} + \xi_2 \frac{\partial f}{\partial y} = \nu (f^N - f).$$

Here, $f(t, \vec{x}, \vec{\xi})$ is the velocity distribution function, $\vec{x} = (x, y)$ is the spatial coordinate vector, $\vec{\xi} = (\xi_1, \xi_2)$ is the molecular velocity vector, ν is the collision frequency and f^N is the equilibrium function depending on model used.

Macroparameters such as the density ρ, the mean velocity $\vec{u}$, the temperature T and others are given by moments of the distribution function f:

$$\rho = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f d\xi_1 d\xi_2 d\xi_3 \,, \quad \rho\vec{u} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \vec{\xi} \cdot f d\xi_1 d\xi_2 d\xi_3 \,, \quad 3\rho RT = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \vec{c}^{\,2} f d\xi_1 d\xi_2 d\xi_3 \,.$$

where $\vec{c} = \vec{\xi} - \vec{u}$ and R is the universal gas constant .

Models

BGK-model. In the BGK model, f^N is a local Maxwellian function,

$$f^N = f^M = \frac{\rho}{(2\pi RT)^{3/2}} \exp\left[-\frac{1}{2RT} \sum_{i=1}^{3} (\xi_i - u_i)(\xi_j - u_j) \right].$$

The main shortcoming of this model is the wrong value of the Prandtl number. It means that Chapman-Enskog expansion of the BGK equation yields the Navier-Stokes equations for a monatomic gas with the Prandtl number $\mathrm{Pr} = 1$ instead of the correct value $\mathrm{Pr} = 2/3$.

S-model. In the Shakhov model, f^N is defined as follows:

$$f^N = \left[1 + \frac{1}{5pRT}(1 - \mathrm{Pr})(\vec{\xi} - \vec{u})\, \vec{c}\, \vec{q} \left(\frac{\vec{c}^{\;2}}{RT} - 5 \right) \right] \cdot f^M \,, \qquad \vec{q} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\vec{c}^{\,2}}{2} \vec{c} f d\xi_1 d\xi_2 d\xi_3.$$

Here $\vec{q}$ is the heat flux vector and $p = \rho RT$ is the pressure.

ES-model. In the ellipsoid statistical model, the equilibrium distribution function f^N is an anisotropic Gaussian:

$$f^N = \frac{\rho}{\pi^{3/2}} \sqrt{\det A} \exp\left[-\sum_{i=1}^{3} A_{ij} (\xi_i - u_i)(\xi_j - u_j) \right] \,,$$

where

$$A_{ij} = \left(\frac{2RT}{\mathrm{Pr}} \delta_{ij} - \frac{2(1 - \mathrm{Pr}) p_{ij}}{\rho \mathrm{Pr}} \right)^{-1} \,, \qquad p_{ij} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (\xi_i - u_i)(\xi_j - u_j) f \xi_1 d\xi_2 d\xi_3.$$

The collision frequency $\nu = p/\mu$ is for both BGK and Shakhov models, where μ is the gas viscosity, while in the ellipsoidal statistical model it is defined as $\nu = \mathrm{Pr} p/\mu$.

Numerical Method

In present paper the relaxation-type kinetic equation is solved using the discrete ordinate method. It implies replacement of exact integration over whole velocity space with approximate integration over finite discrete set of points using an appropriate quadrature. We used the composite Simpson integration rule in our computations. The diffuse reflection boundary conditions are imposed on the plate surface.

Since the equation is solved in the multidimensional phase space, it requires considerable computational efforts (especially at high free-stream Mach numbers when the number of grid points necessary for accurate integration over the velocity space increases significantly). Thus, it is preferable to use a high-order method for spatial approximation. The high resolution shock-capturing WENO scheme of the 5th order [10], which has proven to be a robust and efficient instrument in many CFD problems, is used in our simulations.

Navier-Stokes equations

The compressible Navier-Stokes equations are solved employing the same WENO scheme for calculating the convective fluxes and the 4th order central finite differences for the diffusive terms.

Velocity slip and temperature jump boundary conditions [11] are imposed on the solid wall to take into account rarefaction effects.

$$u_s = \alpha_u \lambda \left(\frac{\partial u}{\partial y}\right)_s, \qquad T_s - T_w = \alpha_T \frac{\gamma}{\gamma - 1} \frac{\lambda}{\mathrm{Pr}} \left(\frac{\partial T}{\partial y}\right)_s \tag{1}$$

Here the subscript s refers to the gas quantities near the wall, u is the velocity component tangential to the wall and T_w is the wall temperature. The boundary conditions (1) can be deduced from an approximate solution of the kinetic equation in the Knudsen layer assuming the diffusive reflection of molecules from the wall with complete accommodation. The calculation gives for the numerical values of coefficients $\alpha_u = 1.142$ and $\alpha_T = 0.5865$ [11].

Direct simulation Monte Carlo method

All DSMC computations are performed using the SMILE software system [12] developed in our laboratory. It is based on the efficient majorant frequency scheme. Two independent grids are employed: the first one in order to organize the particle collisions, and the second one for the sampling of the macroparameters. Both the grids are based on the uniform rectangular background cells, which are split into smaller cells, if necessary. The diffuse reflection model with complete accommodation is employed at the plate surface.

DISCUSSION OF RESULTS

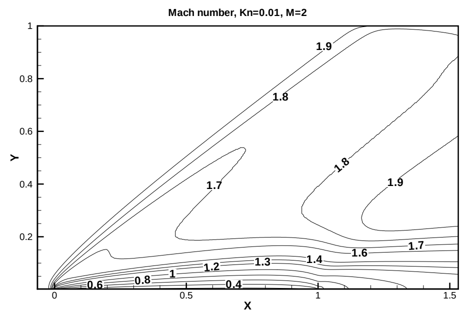

FIGURE 1. Flow past flat plate. Mach number flowfield. $M_\infty = 2$.

The Navier-Stokes and relaxation-type kinetic equation computations were performed on the spatial grid containing 240×480 points. The grid in the velocity space includes from 33×33 up to 127×127 (in some computations at $M_\infty = 10$) points. Thus, the maximum size of the grid in the phase space was about $1.86{\cdot}10^9$ points. In DSMC computations were used from 600 thousands up to 1 million of model particles. The collisional cell size and the time step were chosen certainly smaller than local mean free path and local mean collision time respectively.

The Mach number flowfield obtained at the freestream Mach number $M_\infty = 2$ using the ES-BGK model is shown in Fig. 1. It demonstrates distinctively such flow features as a leading edge shock wave, a thick boundary layer, a wake behind the plate and an expansion fan near the plate trailing edge.

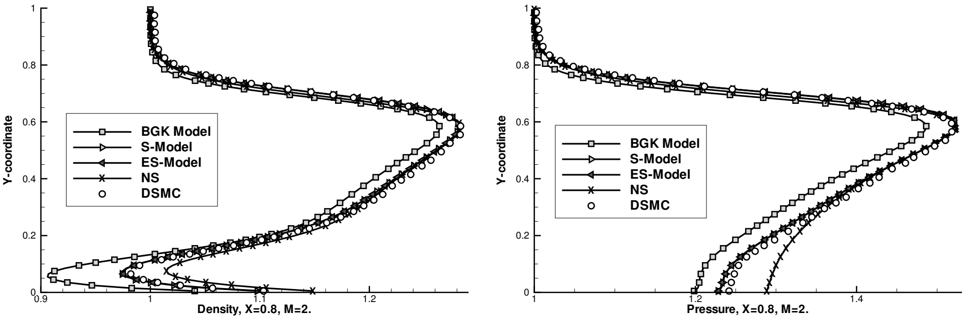

FIGURE 2. Density and pressure profiles in section $X = 0.8$, $M_\infty = 2$.

In Fig. 2 the density and pressure distributions at the cross-section $X = 0.8$ are shown. The Shakhov model and the ellipsoid statistical model show almost indistinguishable profiles which are close to DSMC data. Also, the profiles obtained from the Navier-Stokes equations match those from the DSMC method in general quite well, however, significant discrepancies can be observed close to the plate surface. The BGK model biases from other profiles, although this result is predictable due to a wrong value of the Prandtl number in this model.

Also, the data on surface coefficients were obtained. In Fig. 3 are presented the distributions of pressure coefficient C_p, skin friction coefficient C_f and heat transfer coefficient C_h. As it can be seen, all three approaches show highly consistent results, excluding small vicinity of the sharp leading edge where the Navier-Stokes equations show inadequate results due to strong rarefaction of the flow.

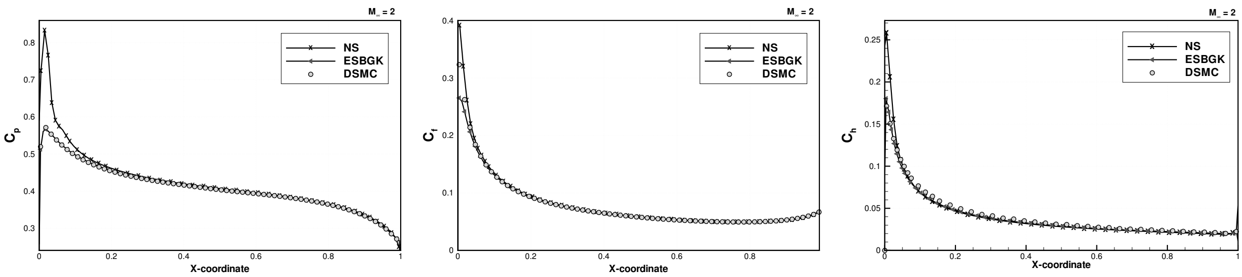

FIGURE 3. Pressure coefficient C_p, skin friction coefficient C_f and heat transfer coefficient C_h, $M_\infty = 2$.

The numerical simulations have been repeated at $M_\infty = 10$ and the results obtained for this hypersonic case are quite different. The BGK model is not used in the calculations because of poor results for lower Mach number flows.

The Shakhov model and the ellipsoid statistical model show relatively good agreement with the DSMC method whereas the Navier-Stokes results differ drastically. Figure 4 shows that in the same section $X = 0.8$ (which is quite far from sharp leading edge) the Navier-Stokes profiles do not coincide even qualitatively with all other approaches.

Similar situation is observed in surface coefficients distributions. The model kinetic equations produce rather accurate results being in reasonable agreement with DSMC data along the entire length of the plate. At the same time solutions of Navier-Stokes equations show big deviation from both the kinetic approaches which decreases gradually to the trailing edge of the plate.

This phenomenon can be explained by a strong violation of equilibrium which occurs in the hypersonic rarefied flow. In Fig. 6 the flowfields of ratio of the longitudinal temperature T_x to the total temperature T computed with the ellipsoidal statistical model are presented. This parameter characterizes the non-equilibrium between different translational degrees of freedom of molecules. As it can be seen in case of the relatively low free-stream Mach number the flow is non-equilibrium only in a small vicinity of the plate leading edge. However, for the hypersonic Mach number, a strong non-equilibrium is observed in an extensive region around the plate.

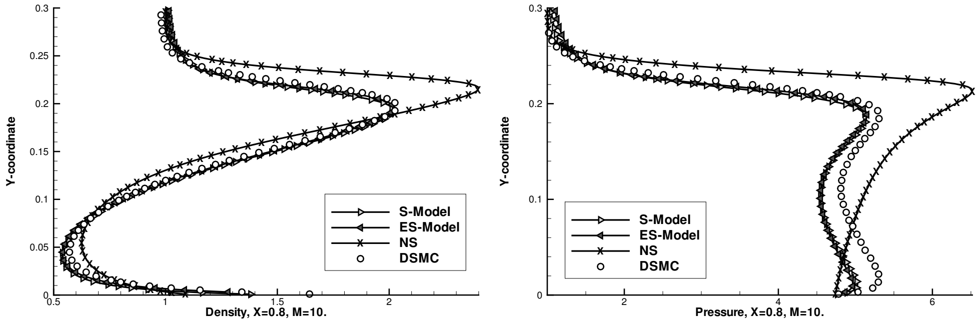

FIGURE 4. Density and pressure profiles in section $X = 0.8$, $M_\infty = 10$.

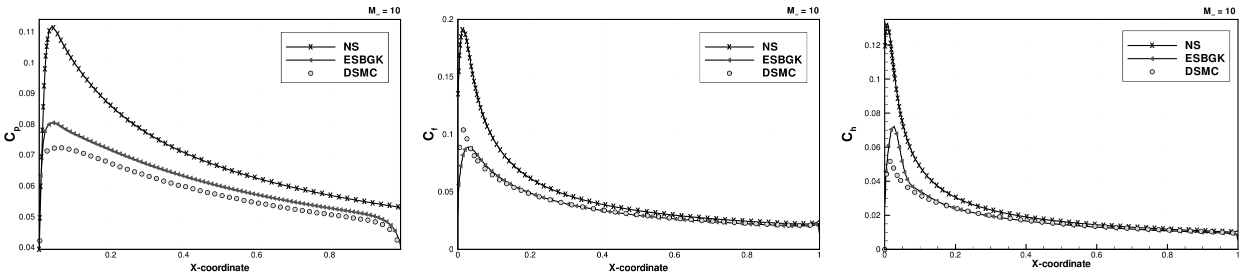

FIGURE 5. Pressure coefficient C_p, skin friction coefficient C_f and heat transfer C_h, $M_\infty = 10$.

Figure 7 shows the slip velocity on the plate surface at $M_\infty = 2$ and 10 computed using different approaches used in the present paper. At $M_\infty = 2$ the ellipsoidal statistical model and the DSMC method give very close results whereas the Navier-Stokes slip velocity u_s calculated using boundary conditions (1) is noticeably different. It should be taken into account, however, that the slip velocity appearing in (1) is a fictitious quantity defined in such a way that the Navier-Stokes solution should coincide with the kinetic solution outside the Knudsen layer rather than the real gas velocity on the wall. Nevertheless, this real gas velocity u_g can be calculated from u_s using an approximate solution of the Boltzmann equation in the Knudsen layer. The calculation leads to the formula [13]

$$u_g = \frac{\sqrt{2/\pi}}{1.1466} u_s = 0.696\, u_s. \tag{2}$$

The quantity recalculated using this formula is also shown in Fig. 7 and denoted there as "NS, gas velocity". It is seen that it is in good agreement with the results of DSMC and ES-BGK computations.

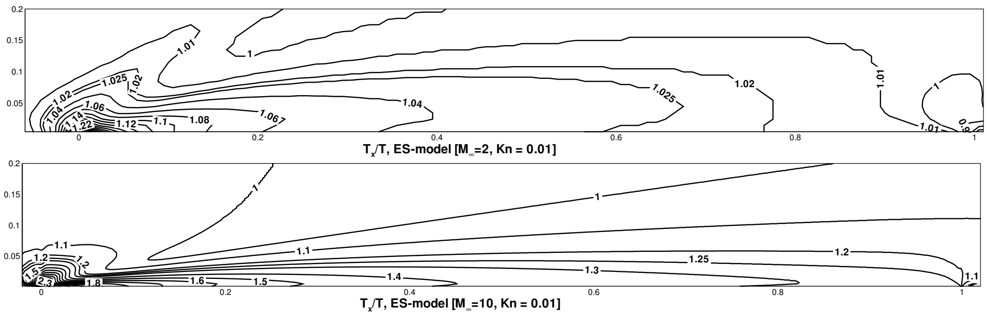

FIGURE 6. Nonequilibrium of temperature, T_x/T. $M_\infty = 2$ (top) and $M_\infty = 10$ (bottom).

Good agreement between DSMC and ES-BGK results persists at $M_\infty = 10$. However, in this case there is a big difference between these results and the Navier-Stokes data and although recalculation using the formula 2 reduces the difference, it does not lead to any reasonable agreement. This once again shows that the Navier-Stokes equations

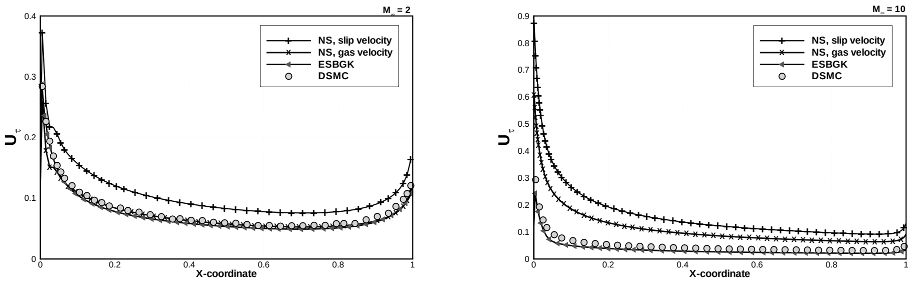

FIGURE 7. Slip velocity on the wall. $M_\infty = 2$ (left) and $M_\infty = 10$ (right).

cannot be used when there is a strong flow non-equilibrium while the description based on model kinetic equation allows us to obtain, in such situations, much more satisfactory results.

CONCLUSION

Numerical simulations of the rarefied flow of a monatomic gas past a finite flat plate have been performed using three different approaches: the Navier-Stokes equations with the velocity slip and temperature jump boundary conditions, Direct Statistical Monte Carlo method and finite difference solving of relaxation-type model kinetic equations using a high-order shock capturing scheme. The computation have been conducted at the free-stream Mach numbers $M_\infty = 2$ and 10 and the Knudsen number $Kn = 0.01$. It has been shown that at $M_\infty = 2$ all three approaches give close results provided that the model equation leads to the correct value of the Prandtl number in the continuum limit. A small difference can be observed only in the immediate vicinity of the plate where the molecular distribution function is far from the equilibrium one. On the contrary, at $M_\infty = 10$ there is no equilibrium between different translational degrees of freedom within rather an extensive region around the plate. As a result, the Navier-Stokes equations fail to predict the flow properties correctly even along with the slip boundary conditions. At the same time, the results obtained from the model kinetic equation are in good agreement with DSMC data. Thus, deterministic solving of the relaxation-type kinetic equations can be successfully used for numerical simulation of non-equilibrium rarefied flows such as the hypersonic flow past bodies with a sharp leading edge.

ACKNOWLEDGMENTS

This work has been supported by the RFBR grant No. 10-08-1203.

REFERENCES

1. P.L. Bhatnagar, E.P. Gross and M. Krook, *Phys. Rev.* **94**, 511–525 (1954)
2. L.H. Holway, *Phys. Fluids* **9**, 1658–1673 (1966)
3. E.M. Shakhov, *Fluid Dynamics* **3**, 95–96 (1968)
4. A.B. Huang and D.L. Hartley, *Phys.Fluids* **12**, 96–108 (1969)
5. A.B. Huang and P.F. Hwang, *Phys.Fluids* **13**, 309–317 (1970)
6. E.M. Shakhov, *Fluid Dynamics* **8**, 275–281 (1973)
7. K. Aoki, K. Kanba and S. Takata, *Phys. Fluids* **9**, 1144–1161 (1997)
8. T.G. Elizarova, I.A.Graur, A. Chpoun and J.C. Lengrand, *Rarefied Gas Dynamics 19* **2**, 780–786. Oxford Univ. Press (1995)
9. I.V. Egorov and A.I. Erofeev, *Fluid Dynamics* **32**, 112–122 (1997)
10. G.-S. Jiang, C.-W. Shu, *J. Comput. Phys* **126**, 202–228 (1996)
11. M.N. Kogan, *Rarefied Gas Dynamics*. Plenum, New York (1969)
12. M.S. Ivanov, G.N. Markelov, S.F. Gimelshein, AIAA Paper 98-2669 (1998)
13. C. Cercignani, *Theory and application of the Boltzmann equation*. Scottish Academic Press, Edinburgh (1975)